**Also solved by Rafik Zeraoulia, Algeria; Michel Bataille, Rouen, France and Arkady Alt, San Jose, California, USA.**

**82.** In descartes co-ordinate system with origin $O$ given are the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a point $M_0$ on it. If $M$ is a point on the ellipse, compute the maximal area of the triangle $OM_0M$.

(BULGARIAN NATIONAL UNIVERSUTY OLYMPIAD 2015)

**Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.**

The answer is $ab/2$. Consider the parametrization $t \mapsto M(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$ of the ellipse. and suppose that $M_0 = M(t_0)$ for some $t_0 \in \mathbb{R}$. Now the area $\mathcal{A}(t)$ of $\triangle OM_0M(t)$ is given by

$$\mathcal{A}(t) = \frac{1}{2}\left|\det(\overrightarrow{OM_0}, \overrightarrow{OM(t)})\right| = \frac{1}{2}\left|\det\begin{bmatrix} a\cos t_0 & a\cos t \\ b\sin t_0 & b\sin t \end{bmatrix}\right| = \frac{ab}{2}|\sin(t-t_0)|$$

Thus the maximum value of $\mathcal{A}(t)$ is $ab/2$ and it is attained when $t = t_0 + \frac{\pi}{2}$.

**Solution 2 by Arkady Alt, San Jose, California, USA.**

Let $M_0\,(x_0, y_0)$ and $M\,(x, y)$ .Then $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2} = 1,\ \ \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and by Cauchy Inequality
we have $[OM_0M] = \frac{1}{2}\left|\det\begin{pmatrix} x & y \\ x_0 & y_0 \end{pmatrix}\right| = \frac{1}{2}|xy_0 - x_0y| = \frac{1}{2}\left|\frac{x}{a}\cdot ay_0 + \frac{y}{b}\cdot(-bx_0)\right| \leq$
$\frac{1}{2}\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\cdot\sqrt{a^2y_0^2+b^2y_0^2} = \frac{1}{2}\sqrt{a^2y_0^2+b^2y_0^2} = \frac{ab}{2}\sqrt{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}} = \frac{ab}{2}.$
Since equality in Cauchy Inequality occurs iff $\left(\frac{x}{a}, \frac{y}{b}\right) = k\,(ay_0, -bx_0)$ then
$x = ka^2y_0, y = -kb^2x_0$ then by replacing $(x, y)$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ with $\left(ka^2y_0, -kb^2x_0\right)$ we obtain $k^2\left(a^2y_0^2+b^2x_0^2\right) = 1 \iff k^2a^2b^2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right) = 1 \implies k = \frac{1}{ab} \implies$
$x = \frac{ay_0}{b}, y = -\frac{bx_0}{a}.$

Since for $(x, y) = \left(\frac{ay_0}{b}, -\frac{bx_0}{a}\right)$ we have $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{\frac{a^2y_0{}^2}{b^2}}{a^2}+\frac{\frac{b^2x_0^2}{a^2}}{b^2} = \frac{y_0^2}{b^2}+\frac{x_0^2}{a^2} = 1$
and $\frac{1}{2}|xy_0 - x_0y| = \frac{1}{2}\left|\frac{ay_0}{b}\cdot y_0 - x_0\cdot\left(-\frac{bx_0}{a}\right)\right| = \frac{ab}{2}\left|\frac{y_0^2}{b^2}+\frac{x_0^2}{a^2}\right| = \frac{ab}{2}$
then $\max\,[OM_0M] = \frac{ab}{2}.$

**83.** Let $P$ be the sum of all $2\times 2$ matrices whose elements are the integers $0, 1, 2$ and $3$, without repetition. Find the matrices:

a) $S = \frac{1}{36}P$;

b) $S^{2015}$;